BRANCHED COVERINGS OF CP^2 AND OTHER BASIC 4-MANIFOLDS

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Abstract

We give necessary and sufficient conditions for a 4-manifold to be a branched covering of CP^2, $S^2 \times S^2$, $S^2 \mathbin{\tilde{\times}} S^2$ and $S^3 \times S^1$, which are expressed in terms of the Betti numbers and the intersection form of the 4-manifold. Moreover, we extend these results to include branched coverings of connected sums of the above manifolds.

Keywords: branched covering, 4-manifold, 2-knot, surface knot.

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Introduction

In [19] the first author proved that any closed orientable PL 4-manifold M is a simple 4-fold covering of S^4 branched over a closed locally flat PL surface self-transversally immersed in S^4. Subsequently, in [11] the self-intersections of the branch surface were shown to be removable once the covering has been stabilized to degree five, obtaining M as a 5-fold covering of S^4 branched over a closed locally flat PL surface embedded in S^4.

On the other hand, it is a classical result of algebraic geometry that any smooth complex algebraic surface $S \subset CP^n$ is a holomorphic branched covering of CP^2, where the branch set is an algebraic curve in CP^2 with nodes and cusps as the only singularities. Furthermore, Auroux in [3] extended this result to all closed symplectic 4-manifolds M, proving that they are realizable as "symplectic" coverings of CP^2 branched over a symplectic surface in CP^2 with nodes and cusps as the only singularities. This means that, up to an integer factor, the symplectic form of M is the lifting of the standard one of CP^2 by such a branched covering.

Hence, it is interesting to study the topology of branched coverings of CP^2, and a natural question is the following: *which closed oriented 4-manifolds are realizable as branched coverings of* CP^2*?*

In this paper we give a complete answer to this question, by proving that a closed connected orientable PL 4-manifold M is a simple branched covering of CP^2 (branched over an embedded locally flat surface) if and only if the second Betti number $b_2(M)$ is positive. In addition, we also characterize the 4-manifolds M that are branched coverings of $S^2 \times S^2$, $S^2 \tilde{\times} S^2$ and $S^3 \times S^1$. Finally, we generalize these results to branched coverings of $\#_m CP^2 \#_n \overline{CP}^2$, $\#_n(S^2 \times S^2)$ and $\#_n(S^3 \times S^1)$.

The proofs of all these results follow the same idea: we split M into two pieces, based on certain submanifolds $N \subset M$, and represent them as branched coverings of standard bounded 4-manifolds by using [20], then we glue such branched coverings together. As a consequence of this argument, we also obtain a representation of the submanifolds $N \subset M$ as branched coverings of suitable standard submanifolds of the base spaces considered above.

For the sake of convenience, we work in the PL category. Nevertheless, our results can be easily translated into the smooth category as well, being PL = Diff in dimension four.

1. Statements

We start by briefly recalling the notion of branched covering, in order to introduce some terminology (see [6] or [10] for more details).

A map $p\colon M \to N$ between compact *oriented* PL manifolds having the same dimension n is called a *branched covering* if and only if it is a non-degenerate *orientation preserving* PL map with the following properties: 1) there is an $(n-2)$-dimensional polyhedral subspace $B_p \subset N$, the *branch set* of p, such that the restriction $p_| : M - p^{-1}(B_p) \to N - B_p$ is an ordinary covering of finite *degree* $d(p)$ (we assume B_p to be minimal with respect to this property); 2) in the bounded case, $p^{-1}(\partial N) = \partial M$ and p preserves the product structure of a collar of the boundaries (which implies that the restriction to the boundary $p_| : \partial M \to \partial N$ is a branched covering of the same degree of p).

Moreover, p is called *simple* if the monodromy of the above mentioned ordinary covering sends any meridian around B_p to a transposition. In this case, also the restriction to the boundary $p_| : \partial M \to \partial N$ is simple.

Now we can state our main theorems, where the following notations are used: CP^2 and $\overline{CP}^2$ for the complex projective space with the standard and the opposite orientation, respectively; $S^2 \tilde{\times} S^2 \cong CP^2 \# \overline{CP}^2$ for the twisted S^2-bundle over S^2; $b_i(M)$ for the i-th Betti number of M;

$$\beta_M : H_2(M)/\operatorname{Tor} H_2(M) \times H_2(M)/\operatorname{Tor} H_2(M) \to \mathbb{Z}$$

for the intersection form of M; and finally $b_2^+(M)$ (resp. $b_2^-(M)$) for the maximal dimension of a vector subspace of $H_2(M;\mathbb{R})$ where β_M is positive (resp. negative) definite (see [10], [12] or [15]).

Theorem 1. *Let M be a closed connected oriented PL 4-manifold. Then, there exists a branched covering $p\colon M \to N$ with:*

(a) $N = CP^2 \Leftrightarrow b_2^+(M) \geq 1$;

(b) $N = \overline{CP}^2 \Leftrightarrow b_2^-(M) \geq 1$;

(c) $N = S^2 \tilde{\times} S^2 \Leftrightarrow b_2^+(M) \geq 1$ *and* $b_2^-(M) \geq 1$;

(d) $N = S^2 \times S^2 \Leftrightarrow b_2^+(M) \geq 1$ *and* $b_2^-(M) \geq 1$;

(e) $N = S^3 \times S^1 \Leftrightarrow b_1(M) \geq 1$.

In all cases, we can assume that p is a simple branched covering of degree $d \leq 4$, whose branch set B_p is a closed locally flat PL surface self-transversally immersed in N. Moreover, B_p can be desingularized to become embedded in N, with the following estimates for the degree d: $d \leq 5$ in cases (a) and (b) for $b_2(M) \geq 2$ and β_M odd, case (c) for β_M odd, case (d) for β_M even, and case (e); $d \leq 6$ in cases (a) and (b) for $b_2(M) \geq 2$ and β_M even, case (c) for β_M even, and case (d) for β_M odd; $d \leq 9$ in cases (a) and (b) for $b_2(M) = 1$.

REMARK 2. If β_M is indefinite, then M is a simple branched covering of all of CP^2, $\overline{CP}^2$, $S^2 \widetilde{\times} S^2$ and $S^2 \times S^2$. On the other hand, if β_M is positive (resp. negative) definite, then among these manifolds CP^2 (resp. $\overline{CP}^2$) is the only one of which M is a branched covering.

For the sake of completeness, we also state the following generalization of Theorem 1. The proof is based on the same methods of that of Theorem 1, and we will only give a sketch of it.

THEOREM 3. *Let M be a closed connected oriented PL 4-manifold and let m and n be non-negative integers. Then, there exists a branched covering $p\colon M \to N$ with:*
(a) $N = \#_m CP^2 \#_n \overline{CP}^2 \Leftrightarrow b_2^+(M) \geq m$ *and* $b_2^-(M) \geq n$;
(b) $N = \#_n(S^2 \times S^2) \Leftrightarrow b_2^+(M) \geq n$ *and* $b_2^-(M) \geq n$;
(c) $N = \#_n(S^3 \times S^1) \Leftrightarrow \pi_1(M)$ *admits a free group of rank n as a quotient.*
In all cases, we can assume that p is a simple branched covering of degree $d \leq 4$, whose branch set B_p is a closed locally flat PL surface self-transversally immersed in N. Moreover, B_p can be desingularized to become embedded in N, with the following estimates for the degree d: $d \leq 5$ in case (a) for $b_2(M) \geq 2(m+n)$ and β_M odd, case (b) for β_M even, and case (c); $d \leq 6$ in case (a) for $b_2(M) \geq 2(m+n)$ and β_M even, and case (b) for β_M odd; $d \leq 9$ in case (a) for $b_2(M) < 2(m+n)$.

We observe that Theorem 3 (a) includes Theorem 1 (a), (b) and (c), being $S^2 \widetilde{\times} S^2 \cong CP^2 \# \overline{CP}^2$. Similarly, it includes the case of $N = \#_m(S^2 \times S^2) \#_n(S^2 \widetilde{\times} S^2)$ with $n \geq 1$, being $(S^2 \times S^2) \# CP^2 \cong (S^2 \widetilde{\times} S^2) \# CP^2$.

The results concerning branched covering representation of submanifolds, which will be obtained in proving the main theorem, as we said in the introduction, can be better stated in light of the next definition.

DEFINITION 4. Let M and N be compact oriented connected n-manifolds, and let $M_1, \ldots, M_k \subset M$ and $N_1, \ldots, N_k \subset N$ be compact locally flat PL oriented submanifolds embedded in M and N, respectively. By a *d-fold branched covering* $p\colon (M; M_1, \ldots, M_k) \to (N; N_1, \ldots, N_k)$ we mean a d-fold branched covering $p\colon M \to N$ whose branch set is transversal to all the submanifolds N_i and such that $p(M_i) = N_i$ and $p_i = p_{|M_i}\colon M_i \to N_i$ preserves the orientation for every $i = 1, \ldots, k$.

Notice that, if p is a (simple) d-fold branched covering as in the definition, then each restriction $p_i\colon M_i \to N_i$ is a (simple) d_i-fold branched covering for some $d_i \leq d$.

Given two closed oriented locally flat PL surfaces $F_1, F_2 \subset M$ in the closed oriented PL 4-manifold M, we will denote by $F_1 \cdot F_2$ their algebraic intersection, that is the number $\beta_M([F_1], [F_2]) \in \mathbb{Z}$.

THEOREM 5. *Let M be a closed connected oriented PL 4-manifold and $F \subset M$ be a closed connected oriented locally flat PL surface. If $d = |F \cdot F| \geq 4$, then there exists a simple d-fold branched covering:*

(a) $p\colon (M; F) \to (CP^2; CP^1)$ *if $F \cdot F$ is positive;*

(b) $p\colon (M; F) \to (\overline{CP}^2; CP^1)$ *if $F \cdot F$ is negative.*

In both cases, $F = p^{-1}(CP^1)$, and B_p is a closed locally flat PL surface self-transversally immersed (embedded for $d \geq 5$) in CP^2 or $\overline{CP}^2$.

THEOREM 6. *Let M be a closed connected oriented PL 4-manifold and $F_1, F_2 \subset M$ be two closed connected oriented locally flat PL surfaces transversal to each other, whose all intersection points are positive. If $F_1 \cdot F_1 = n\,d$, $F_1 \cdot F_2 = d$ and $F_2 \cdot F_2 = 0$ for some integers n and $d \geq 4$, then there exists a simple d-fold branched covering:*

(a) $p\colon (M; F_1, F_2) \to (S^2 \times S^2; S^2_1, S^2_2)$, *with S^2_1 and S^2_2 respectively a section with self-intersection n and a fiber of the trivial bundle $S^2 \times S^2 \to S^2$, if n is even;*

(b) $p\colon (M; F_1, F_2) \to (S^2 \mathbin{\tilde\times} S^2; S^2_1, S^2_2)$, *with S^2_1 and S^2_2 respectively a section with self-intersection n and a fiber of the twisted bundle $S^2 \mathbin{\tilde\times} S^2 \to S^2$, if n is odd.*

In both cases, $F_i = p^{-1}(S^2_i)$, and B_p is a closed locally flat PL surface self-transversally immersed (embedded for $d \geq 5$) in $S^2 \times S^2$ or $S^2 \mathbin{\tilde\times} S^2$.

We observe that a section as specified in the above statement exists for every integer n. In fact, given two copies of the trivial bundle $B^2 \times S^2 \to B^2$, we can glue them along the boundary by the map $(\alpha, x) \mapsto (\alpha, \rho_{n\alpha}(x))$, with $\rho_\alpha\colon R^3 \to R^3$ the rotation of α radians around the third axis. In this way, we get the trivial bundle $S^2 \times S^2 \to S^2$ or the twisted bundle $S^2 \mathbin{\tilde\times} S^2 \to S^2$, depending on the parity of n, with two natural sections deriving from the two copies of $B^2 \times \{(0, 0, \pm 1)\}$, both having self-intersection n.

THEOREM 7. *Let M be a closed connected oriented PL 4-manifold and $N \subset M$ be a closed connected oriented (locally flat) PL 3-manifold. For any $d \geq 4$ there exists a simple d-fold branched covering:*

(a) $p\colon (M; N) \to (S^4; S^3)$ *if N disconnects M;*

(b) $p\colon (M; N) \to (S^3 \times S^1; S^3 = S^3 \times \{*\})$ *if N does not disconnect M.*

In both cases, $N = p^{-1}(S^3)$, and B_p is a closed locally flat PL surface self-transversally immersed (embedded for $d \geq 5$) in S^4 or $S^3 \times S^1$.

Our last result is not related to the main theorem. Still, we include it for the sake of completeness, since it provides a representation of surfaces in 4-manifolds as branched covering of trivial 2-spheres in S^4 (cf. [17] for links in 3-manifolds).

THEOREM 8. *Let M be a closed connected oriented PL 4-manifold and $F \subset M$ be a closed oriented locally flat PL surface with k connected components $F_1, \dots, F_k$, such that $F_i \cdot F_i = 0$ for every $i = 1, \dots, k$ (that is, the normal bundle ν_F is trivial). Then, for any $d \geq 4$ there is a simple d-fold branched covering $p\colon (M; F) \to (S^4; T_k)$, with $T_k \subset S^4$ the trivial 2-link with k spherical components and $B_p \subset S^4$ a closed locally flat PL surface self-transversally immersed (embedded for $d \geq 5$) in S^4, which is transversal to T_k. Moreover, p can be chosen in such a way that each restriction $p_{|F_i}\colon F_i \to p(F_i) \cong S^2$ is equivalent to any given simple branched covering of degree $d_i \leq d - 2$. In particular, if F is consists of 2-spheres, we can assume $B_p \cap T_k = \emptyset$, hence p is the trivial d-fold covering over T_k.*

We observe that any closed oriented locally flat PL surface $F \subset S^4$ admits a branched covering representation as in the theorem.

2. Proofs

First of all we need the following technical definition.

DEFINITION 9. A simple branched covering $p\colon M \to S^3$ is said to be *ribbon fillable* if it can be extended to a simple branched covering $q\colon W \to B^4$ whose branch set $B_q \subset B^4$ is a ribbon surface (which immediately implies that $M = \partial W$, $B_p = \partial B_q \subset S^3$ is a link, and $d(p) = d(q)$). For the sake of convenience, we also call ribbon fillable any simple branched cover $p\colon M \to S^3_1 \cup \ldots \cup S^3_k$ that is a disjoint union of ribbon fillable coverings.

We recall that any compact connected oriented 4-dimensional 2-handlebody W is a simple 3-fold covering of B^4 branched over a ribbon surface in B^4 (see [16]), and that any closed connected oriented 3-manifold M can be realized as the boundary ∂W of such a handlebody W. More precisely, a ribbon fillable simple 3-fold branched covering $p\colon M \to S^3$ can be constructed starting from any Kirby diagram that provides an integral surgery presentation of M (see [6, 7]).

Up to *covering stabilization*, a (simple) d-fold branched covering over S^n or B^n can modified into a (simple) branched covering having an arbitrary degree $d' > d$. This changes the branch set by the addition of $d' - d$ separate flat $(n-2)$-spheres in S^n or $(n-2)$-balls in B^n, with monodromies $(d\,d{+}1), \ldots, (d'{-}1\,d')$. Then, any stabilization of a ribbon fillable covering $p\colon M \to S^3$ is still ribbon fillable.

The proof of all the results stated in the previous section depend on the following theorem, which was established in [20].

THEOREM 10 ([20]). *Let M be a compact connected oriented PL 4-manifold whose boundary has k connected components, and let $B^4_1, \ldots, B^4_k \subset S^4$ be a collection of pairwise disjoint PL 4-balls bounded by the 3-spheres $S^3_1, \ldots, S^3_k \subset S^4$, respectively. Any d-fold ribbon fillable simple branched covering $p\colon \partial M \to S^3_1 \cup \ldots \cup S^3_k$ of degree $d \geq 4$, extends to a simple d-fold covering $q : M \to S^4 - \mathrm{Int}(B^4_1 \cup \ldots \cup B^4_k)$ such that B_q is a locally flat self-transversal PL surface properly immersed (embedded for $d \geq 5$) in $S^4 - \mathrm{Int}(B^4_1 \cup \ldots \cup B^4_k)$.*

Given a closed connected oriented surface F, we denote by $\xi_{F,e} : D_e(F) \to F$ the oriented disk bundle over F of Euler number $e \in \mathbb{Z}$. In proving Theorems 5 and 6, we will use the trivial observation formalized in the next lemma.

LEMMA 11. *If $p\colon F \to G$ is a (simple) branched covering of degree $d \geq 1$ between closed connected oriented surfaces, then the pullback $p^*(\xi_{G,e})$ is bundle equivalent to $\xi_{F,de}$ for every $e \in \mathbb{Z}$. Moreover, p lifts to a (simple) branched covering $q\colon D_{de}(F) \to D_e(G)$ having the same degree d and branch set $B_q = \xi^{-1}_{G,e}(B_p)$.*

Now, we prove Theorems 5, 6 and 7 and then derive Theorem 1 from them.

Proof of Theorem 5. Case (b) immediately follows from case (a) by reversing the orientation of M. So, it suffices to prove case (a), supposing $d = F \cdot F \geq 4$.

Let $T_F \subset M$ be a tubular neighborhood of F in M, and $T_{CP^1} \subset CP^2$ be a tubular neighborhood of CP^1 in CP^2. Then, given any simple d-fold branched covering

$f : F \to S^2$ and taking into account the PL homeomorphisms $T_F \cong D_{F,d}$ and $T_{CP^1} \cong D_{CP^1,1} \cong D_{S^2,1}$, we can apply Lemma 11 to obtain an induced simple d-fold branched covering $t\colon T_F \to T_{CP^1}$. In particular, we assume f to be the d-fold stabilization of a 2-fold covering. In this case, B_f consists of $2(g(F)+d-1)$ points with monodromies $(1\,2),(1\,2),\dots,(1\,2),(1\,2),(2\,3),(2\,3),\dots,(d{-}1\,d),(d{-}1\,d)$, and the branch set B_t consists of $2(g(F)+d-1)$ disks with the same monodromies.

Now, we set $W = \mathrm{Cl}(M - T_F)$ and $Y = \mathrm{Cl}(CP^2 - T_{CP^1}) \cong B^4$. Therefore, the restriction $t_{|\partial}\colon \partial T_F \to \partial T_{CP^1}$, can also be thought of as a simple d-fold covering $t_{|\partial}\colon \partial W \to S^3$ branched over $2(g(F)+d-1)$ fibers of the Hopf fibration $S^3 \to S^2$ with the above listed monodromies. As such, $t_{|\partial}$ is ribbon fillable to a simple covering of B^4 branched over $g(F)+d-1$ linked and twisted ribbon annuli with monodromies $(1\,2),\dots,(1\,2),(2\,3),\dots,(d{-}1\,d)$. Then, Theorem 10 allows us to extend $t_{|\partial}$ to a simple d-fold covering $q\colon W \to Y$ branched over a self-transversally immersed (embedded for $d \geq 5$) surface.

Finally, we can define the desired covering p as the union of the coverings t and q, which share the same restriction to the boundary. Namely, $p = t \cup_\partial q\colon M = T_F \cup_\partial W \to CP^2 = T_{CP^1} \cup_\partial Y$. $\square$

Proof of Theorem 6. For the sake of convenience, we denote by $\xi\colon X \to S^2$ the trivial bundle $S^2 \times S^2 \to S^2$ or the twisted bundle $S^2 \tilde{\times} S^2 \to S^2$, depending on whether n is even or odd.

Let $x_1,\dots,x_d \in M$ be the points of $F_1 \cap F_2$. For each $i = 1,\dots,d$, let $U_i \subset F_1$ and $V_i \subset F_2$ be disk neighborhoods of x_i in F_1 and F_2, respectively, such that $U_i \cap U_j = V_i \cap V_j = \emptyset$ if $i \neq j$. Consider tubular neighborhoods $T_{F_1}, T_{F_2} \subset M$ of F_1 and F_2, respectively, in M. Then, there are PL homeomorphisms $\tau_{F_1} : T_{F_1} \to D_{F_1,nd}$ and $\tau_{F_2}\colon T_{F_2} \to D_{F_2,0} \cong F_2 \times B^2$ that canonically identify F_1 and F_2 with the 0-section of $\xi_{F_1,nd}$ and $\xi_{F_2,0}$, respectively. We can arrange T_{F_1} and T_{F_2} in such a way that $\tau_{F_1}(T_{F_1} \cap T_{F_2}) = \xi^{-1}_{F_1,nd}(U_1 \cup \ldots \cup U_d)$ and $\tau_{F_2}(T_{F_1} \cap T_{F_2}) = \xi^{-1}_{F_2,0}(V_1 \cup \ldots \cup V_d)$, hence $T_{F_1} \cap T_{F_2} \cong (U_1 \times V_1) \cup \ldots \cup (U_d \times V_d)$. This implies that $T_{F_1} \cup T_{F_2} \subset M$ is a regular neighborhood of $F_1 \cup F_2$ in M.

Similar data $x, U, V, T_{S^2_1}, T_{S^2_2}, \tau_{S^2_1}$ and $\tau_{S^2_2}$ can be considered for the 2-spheres S^2_1 and S^2_2 in X. More precisely: x is the unique point of $S^2_1 \cap S^2_2$, $U \subset S^2_1$ and $V \subset S^2_2$ are disk neighborhoods of x in S^2_1 and S^2_2, respectively; $T_{S^2_1}, T_{S^2_2} \subset X$ are tubular neighborhoods of S^2_1 and S^2_2, respectively; $\tau_{S^2_1} : T_{S^2_1} \to D_{S^2_1,n}$ and $\tau_{S^2_2} : T_{S^2_2} \to D_{S^2_2,0} \cong S^2_2 \times B^2$, are PL homeomorphisms that canonically identify S^2_1 and S^2_2 with the 0-section of $\xi_{S^2_1,n}$ and $\xi_{S^2_2,0}$, respectively; $\tau_{S^2_1}(T_{S^2_1} \cap T_{S^2_2}) = \xi^{-1}_{S^2_1,n}(U)$ and $\tau_{S^2_2}(T_{S^2_1} \cap T_{S^2_2}) = \xi^{-1}_{S^2_2,0}(V)$. Hence, we have $T_{S^2_1} \cap T_{S^2_2} \cong U \times V$, and $T_{S^2_1} \cup T_{S^2_2}$ is a regular neighborhood of $S^2_1 \cup S^2_2$ in X.

Now, let $f_1\colon F_1 \to S^2_1$ and $f_2\colon F_2 \to S^2_2$ be simple d-fold branched coverings, both realized as the d-fold stabilization of a 2-fold covering, like in the proof of Theorem 5. Up to the PL homeomorphisms $\tau_{S^2_1}$ and $\tau_{S^2_2}$, we can apply Lemma 11 to get the induced simple d-fold branched coverings $t_1 : T_{F_1} \to T_{S^2_1}$ and $t_2 : T_{F_2} \to T_{S^2_2}$. As it can be easily realized, t_1 and t_2 can be arranged in such a way that their restrictions over $U \times V$ both coincide with the same trivial ordinary covering $(U_1 \times V_1) \cup \ldots \cup (U_d \times V_d) \to U \times V$, up to the above identifications via the homeomorphisms $\tau_{S^2_1}$ and $\tau_{S^2_2}$ in the base spaces, and τ_{F_1} and τ_{F_2} in the total spaces.

This allows us to consider their union $t = t_1 \cup t_2 : T_{F_1} \cup T_{F_2} \to T_{S^2_1} \cup T_{S^2_2}$, which still is a simple d-fold branched covering.

In the usual Kirby diagram representation of X, with two 2-handles attached to B^4 along the components of the Hopf link, one with framing n to give $T_{S^2_1}$ and the other with framing 0 to give $T_{S^2_2}$, the branch set B_t consists of $2(g(F_1)+d-1)$ disks parallel to the co-core of the former 2-handle and $2(g(F_2)+d-1)$ disks parallel to the co-core of the latter 2-handle. The disks of each family have the same monodromy in pairs.

Looking at the complement of the considered tubular neighborhoods, we put $W = \mathrm{Cl}(M-(T_{F_1}\cup T_{F_2}))$ and $Y = \mathrm{Cl}(X-(T_{S^2_1}\cup T_{S^2_2})) \cong B^4$. The restriction of t to the boundary gives a simple d-fold branched covering $t_{|\partial} : \partial W \to \partial Y \cong S^3$. In the above mentioned Kirby diagram representation of X, the branch set $B_{t_{|\partial}} = B_{t_{1|\partial}} \cup B_{t_{2|\partial}}$ consists of $2(g(F_1) + d - 1)$ "parallel" copies of the belt sphere of the n-framed 2-handle and $2(g(F_2) + d - 1)$ "parallel" copies of the belt sphere of the 0-framed 2-handle, as shown in the left side of Figure 1. Then, $t_{|\partial}$ is ribbon fillable to a simple covering of B^4 branched over the ribbon surface described on the right side of Figure 1. So, we can use Theorem 10 for extending $t_{|\partial}$ to a simple d-fold covering $q : W \to Y$ branched over a self-transversally immersed (embedded for $d \geq 5$) surface, and conclude the proof by putting $p = t \cup_\partial q : M = (T_{F_1} \cup T_{F_2}) \cup_\partial W \to X = (T_{S^2_1} \cup T_{S^2_2}) \cup_\partial Y$. □

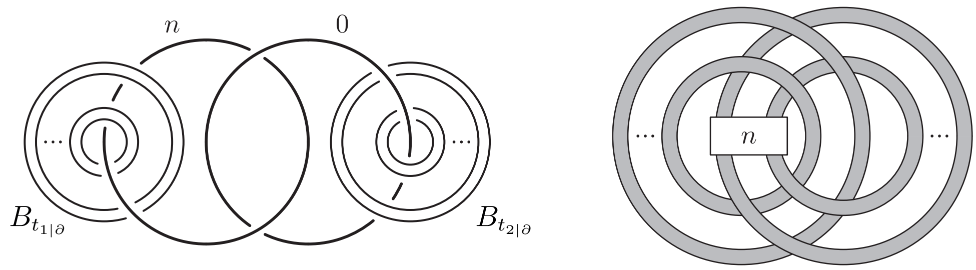

FIGURE 1. Ribbon fillability of $t_{|\partial}$.

Proof of Theorem 7. According to what we said at the beginning of this section, there exists a ribbon fillable d-fold branched covering $c : N \to S^3$.

If N disconnects M, let $M_1, M_2 \subset M$ be the closures of the two connected components of $M - N$. Then, M_1 and M_2 are two PL compact oriented 4-manifolds with $\partial M_1 = \partial M_2 = N$, such that $M = M_1 \cup M_2$. By Theorem 10, the branched covering c extends to two simple d-fold branched coverings $p_1 : M_1 \to S^4_-$ and $p_2 : M_2 \to S^4_+$, both branched over a locally flat PL surface self-transversally immersed (embedded if $d \geq 5$) in the base space, where we denote by $S^n_\pm \subset S^n$ the two hemispheres bounded by $S^{n-1} \subset S^n$. Therefore, we can put $p = p_1 \cup p_2 : M \to S^4$.

In the case where N does not disconnect M, we consider the decomposition $M = M_1 \cup M_2$, with M_1 a collar of N in M, and $M_2 = \mathrm{Cl}(M - M_1)$. The simple d-fold covering $p_1 = c \times \mathrm{id}_{S^1_-} : M_1 \cong N \times S^1_- \to S^3 \times S^1_-$ is branched over the locally flat PL surface $B_c \times S^1_-$, which is properly embedded in $S^3 \times S^1_-$. The restriction of p_1 to the boundary is a ribbon fillable d-fold branched covering $\partial M_1 = \partial M_2 \to \partial(S^3 \times S^1_-) = \partial(S^3 \times S^1_+)$, which by Theorem 10 admits a simple d-fold extension

$p_2 : M_2 \to S^3 \times S^1_+$ branched over a locally flat PL surface self-transversally immersed (embedded if $d \geq 5$) in $S^3 \times S^1_+$. So, also in this case we can conclude by putting $p = p_1 \cup p_2 : M \to S^3 \times S^1$. $\square$

Proof of Theorem 1. First of all, we recall the well known fact that in a closed connected oriented PL 4-manifold M any homology class $\alpha \in H_2(M)$ can be represented by a closed oriented locally flat PL surface $F \subset M$ (see [10] or [12]). Moreover, F can be easily made connected by embedded surgery. Similarly, any homology class $\alpha \in H_3(M)$ can be represented by a closed oriented locally flat PL 3-manifold $N \subset M$, but in this case N can be made connected only if α is primitive (see [14]).

(a). Given any d-fold branched covering $p\colon M \to CP^2$, we can assume up to PL isotopy that $B_p \subset CP^2$ meets CP^1 transversally. Then, $F = p^{-1}(CP^1) \subset M$ is a closed oriented locally flat PL surface, which represents a non-null element $\varphi \in H_2(M)/\operatorname{Tor} H_2(M)$ such that $\beta_M(\varphi,\varphi) = d > 0$. Hence, $b_2(M) \geq 1$ and β_M is not negative definite.

For the converse, assume that $b_2(M) \geq 1$ and β_M is not negative definite. If β_M is odd, then it is diagonalizable. This follows by a theorem of Donaldson for definite intersection forms of closed oriented PL 4-manifolds [9], while it is a general fact for odd indefinite unimodular forms [15]. Hence, there exists $\delta_1 \in H_2(M)/\operatorname{Tor} H_2(M)$ such that $\beta_M(\delta_1,\delta_1) = 1$, and for $b_2(M) \geq 2$ there exists also $\delta_2 \in H_2(M)/\operatorname{Tor} H_2(M)$ such that $\beta_M(\delta_1,\delta_2) = 0$ and $\beta_M(\delta_2,\delta_2) = \pm 1$. Otherwise, if β_M is even, then, again by Donaldson's theorem [9], it is indefinite, and so it contains a hyperbolic direct summand (see [15], [10] or [12]). For a basis $\eta_1, \eta_2 \in H_2(M)/\operatorname{Tor} H_2(M)$ of such subspace, we have $\beta_M(\eta_1,\eta_1) = \beta_M(\eta_2,\eta_2) = 0$ and $\beta_M(\eta_1,\eta_2) = 1$. In both cases, there exists $\varphi \in H_2(M)/\operatorname{Tor} H_2(M)$ such that $\beta_M(\varphi,\varphi) = 4$, with $\varphi = 2\delta_1$ for β_M odd, and $\varphi = \eta_1 + 2\eta_2$ for β_M even.

The desired branched covering $p\colon M \to CP^2$ can be obtained by applying Theorem 5 (a) to $F \subset M$, with F any closed connected oriented locally flat PL surface that represents the homology class φ, and $d = 4$. In this way, the branch set B_p turns out to be a closed locally flat PL surface self-transversally immersed in CP^2.

To obtain an embedded branch set, we can apply Theorem 5 (a) with F representing a suitable different class φ'. Namely: $\varphi' = 3\delta_1$, giving $d = 9$, if $b_2(M) = 1$; $\varphi' = (2 - \beta_M(\delta_2,\delta_2))\,\delta_1 + 2\delta_2$, giving $d = 5$, if $b_2(M) \geq 2$ and β_M is odd; $\varphi' = \eta_1 + 3\eta_2$, giving $d = 6$, if $b_2(M) \geq 2$ and β_M is even.

(b). This case immediately follows from case (a), by reversing the orientations.

(c) and (d). Denote by $\xi : X \to S^2$ the bundle $S^2 \times S^2 \to S^2$ or $S^2 \mathbin{\widetilde{\times}} S^2 \to S^2$, depending on the case, and let S^2_1, $S^2_2 \subset X$ be any PL section and fiber of ξ, respectively. Given a branched d-fold covering $p\colon M \to X$, we can assume up to PL isotopy that $B_p \subset X$ meets both the surfaces S^2_1 and S^2_2 transversally. Then, $F_1 = p^{-1}(S^2_1) \subset M$ and $F_2 = p^{-1}(S^2_2) \subset M$ are closed oriented locally flat PL surfaces such that $F_1 \cdot F_2 = d > 0$ and $F_2 \cdot F_2 = 0$. It follows that the homology class $\varphi \in H_2(M)/\operatorname{Tor} H_2(M)$ represented by F_2 is non-zero and $\beta_M(\varphi,\varphi) = 0$. Therefore, β_M is indefinite, and so $b_2(M) \geq 2$.

Vice versa, assuming $b_2(M) \geq 2$ and β_M indefinite, we want to construct a branched covering $p\colon M \to X$ by applying Theorem 6 to a pair $F_1, F_2 \subset M$ of closed connected oriented locally flat PL surfaces, with F_i representing a suitable homology class $\varphi_i \in H_2(M)/\operatorname{Tor} H_2(M)$. The choice of such classes will depend on whether X is $S^2 \times S^2$ or $S^2 \widetilde{\times} S^2$, β_M is odd or even, and we insist that the branch set B_p is an embedded surface or not. In all the cases, $F_1 \cdot F_1 = nd$, $F_1 \cdot F_2 = d$ and $F_2 \cdot F_2 = 0$, with $n = 0$ for $X = S^2 \times S^2$ or $n = 1$ for $S^2 \widetilde{\times} S^2$, and $d = 4$ if transversal self-intersection are admitted for B_p, while $d = 5$ or 6 if B_p is required to be embedded. Once the two classes φ_1 and φ_2 are chosen, we can isotope F_2 to meet F_1 transversally, and then an embedded surgery can be performed, without changing the homology classes, to eliminate each pair of opposite intersection points (if any) between F_1 and F_2. This determines new surfaces, which we still denote by F_1 and F_2, with d transversal positive intersection points, as in the hypotheses of Theorem 6.

We are left to choose the classes φ_1 and φ_2. If β_M is odd, then it is diagonalizable and so there exist $\delta_1, \delta_2 \in H_2(M)/\operatorname{Tor} H_2(M)$ such that $\beta_M(\delta_1, \delta_1) = 1$, $\beta_M(\delta_1, \delta_2) = 0$, and $\beta_M(\delta_2, \delta_2) = -1$. Then, we get: $d = 4$ and $n = 0$, for $\varphi_1 = \delta_1 + \delta_2$ and $\varphi_2 = 2(\delta_1 - \delta_2)$; $d = 6$ and $n = 0$, for $\varphi_1 = \delta_1 + \delta_2$ and $\varphi_2 = 3(\delta_1 - \delta_2)$; $d = 4$ and $n = 1$, for $\varphi_1 = 2\delta_1$ and $\varphi_2 = 2(\delta_1 - \delta_2)$; $d = 5$ and $n = 1$, for $\varphi_1 = 3\delta_1 + 2\delta_2$ and $\varphi_2 = \delta_1 - \delta_2$. If instead β_M is even, there exist $\eta_1, \eta_2 \in H_2(M)/\operatorname{Tor} H_2(M)$ such that $\beta_M(\eta_1, \eta_1) = \beta_M(\eta_2, \eta_2) = 0$ and $\beta_M(\eta_1, \eta_2) = 1$ (see the analogous case in the proof of (a) above). In this case, we get: $d = 4$ and $n = 0$, for $\varphi_1 = \eta_1$ and $\varphi_2 = 4\eta_2$; $d = 5$ and $n = 0$, for $\varphi_1 = \eta_1$ and $\varphi_2 = 5\eta_2$; $d = 4$ and $n = 1$, for $\varphi_1 = \eta_1 + 2\eta_2$ and $\varphi_2 = 4\eta_2$; $d = 6$ and $n = 1$, for $\varphi_1 = \eta_1 + 3\eta_2$ and $\varphi_2 = 6\eta_2$.

(e). Given any d-fold branched covering $p\colon M \to S^3 \times S^1$, we can assume up to PL isotopy that $B_p \subset S^3 \times S^1$ meets $S^3 \times \{*\}$ transversally and is disjoint from $\{*\} \times S^1$. Then, $N = p^{-1}(S^3 \times \{*\}) \subset M$ and $C = p^{-1}(\{*\} \times S^1) \subset M$ are closed oriented locally flat PL submanifolds of dimensions 3 and 1, respectively, such that $N \cdot C = d > 0$. Then, C represents a non-trivial homology class in $H_1(M)/\operatorname{Tor} H_1(M)$, and so $b_1(M) \geq 1$.

Vice versa, for $b_1(M) = b_3(M) \geq 1$, let $N \subset M$ be a closed connected oriented locally flat 3-manifold representing a primitive non-trivial element of $H_3(M)$. Then N does not disconnect M and we can apply Theorem 7 (b) to get the desired branched covering $p\colon M \to S^3 \times S^1$. $\square$

Proof of Theorem 3. We only sketch the proof, because it follows the same ideas of the proof of Theorem 1. For items (a) and (b) the implications to the right are straightforward, so we only discuss the implications to the left.

(a). We claim that there exists a sublattice $\Lambda_{m,n}(k) \subset (H_2(M)/\operatorname{Tor} H_2(M), \beta_M)$ which is isomorphic to $\oplus_m \langle k \rangle \oplus_n \langle -k \rangle$ for a certain $k \in \mathbb{N}$ that will be specified below, where $\langle k \rangle$ denotes the integral rank 1 lattice of determinant k.

Indeed, if β_M is odd, then the lattice $(H_2(M)/\operatorname{Tor} H_2(M), \beta_M)$ is isomorphic to $\oplus_{b_2^+} \langle 1 \rangle \oplus_{b_2^-} \langle -1 \rangle$, with $b_2^\pm = b_2^\pm(M)$ and $\Lambda_{m,n}(4)$ can be obtained from such decomposition. Moreover, one can obtain $\Lambda_{m,n}(5)$ if $b_2(M) \geq 2(m+n)$ by using the extra generators as in the proof of Theorem 1, or $\Lambda_{m,n}(9)$ if $b_2(M) < 2(m+n)$ by taking the triples of the generators.

If β_M is even, then the lattice $(H_2(M)/\operatorname{Tor} H_2(M), \beta_M)$ is isomorphic to $\oplus_a(\pm E_8) \oplus_b H$ for some $a, b \in \mathbb{N}$ with $b \geq 1$, where E_8 is the symmetric rank 8 positive definite indecomposable unimodular lattice and H is the hyperbolic rank 2 lattice.

With respect to a suitable basis, E_8 can be represented by the matrix

$$A_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

In this basis, the sublattice of E_8 spanned by the columns $g_1, \dots, g_8$ of the matrix

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -2 & -2 & -4 \\ 0 & 1 & 0 & 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & -2 & -2 & -4 & -6 \\ 0 & 0 & 1 & 0 & 1 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 3 \end{pmatrix}$$

is isomorphic to $\oplus_8\langle 2\rangle$. We then obtain $\oplus_8\langle 4\rangle \subset E_8$ as the sublattice spanned by all vectors of the form $g_{2i-1} \pm g_{2i}$ for $i \in \{1, 2, 3, 4\}$.

Moreover, we obtain $\oplus_8\langle 6\rangle \subset E_8$ as the sublattice spanned by all vectors of the form

$$\begin{aligned} &g_{i+1} + g_{i+2} - g_{i+3}, \quad g_{i+1} - g_{i+2} + g_{i+4} \\ &g_{i+1} + g_{i+3} - g_{i+4}, \quad g_{i+2} + g_{i+3} + g_{i+4} \end{aligned}$$

for $i \in \{0, 4\}$.

On the other hand, we can find sublattices $\oplus_2\langle k\rangle \subset H$ for $k = 4, 6$. Therefore, the lattice $(H_2(M)/\operatorname{Tor} H_2(M), \beta_M)$ with β_M even, contains a sublattice isomorphic to $\oplus_{b_2^+}\langle k\rangle \oplus_{b_2^-}\langle -k\rangle$ for $k = 4, 6$, from which we get a sublattice $\Lambda_{m,n}(k)$ for $k = 4, 6$.

Now, we consider the proper sublattice $\Lambda_{m,n}(k)$ according to the particular case of item (a) that we want to prove, and represent the generators of $\Lambda_{m,n}(k)$ by disjoint embedded oriented connected PL locally flat surfaces $F_1, \dots, F_{m+n} \subset M$. We also consider $CP^1_1, \dots, CP^1_{m+n} \subset N$, where CP^1_i is a projective line in the i-th connected summand of $N = \#_m CP^2 \#_n \overline{CP}^2$.

Then, we construct k-fold simple branched coverings $t_i \colon T_{F_i} \to T_{CP^1_i}$ as in the proof of Theorem 5, whose restrictions on the boundary are ribbon fillable. We put $W = \operatorname{Cl}(M - \cup_i T_{F_i})$ and $Y = \operatorname{Cl}(N - \cup_i T_{CP^1_i}) \cong \#_{m+n} B^4 \cong S^4 - \operatorname{Int}(B^4_1 \cup \dots \cup B^4_{m+n})$ and by Theorem 10 we get a simple branched covering $p \colon M \to N$ as desired.

(b). It is enough to observe that for β_M even there are at least n hyperbolic direct summands in the intersection lattice of M, while for β_M odd an argument similar to that in the proof of Theorem 1 gives suitable homology classes that can be used to construct the branched covering that satisfies the statement.

(c). We denote by $\mathbb{F}_n$ the free group of rank n. Suppose that there is a d-fold branched covering $p\colon M \to N = \#_n(S^3 \times S^1)$ for some $d \geq 1$. Let $\gamma_1, \dots, \gamma_n \in \pi_1(N) \cong \mathbb{F}_n$ be the generators. By lifting loops, we can find elements $\tilde{\gamma}_1, \dots, \tilde{\gamma}_n \in \pi_1(M)$ such that $p_*(\tilde{\gamma}_i) = \gamma_i^{a_i}$ for certain $a_i \in \{1, \dots, d\}$ and for all $i = 1, \dots, n$, where $p_*\colon \pi_1(M) \to \pi_1(N)$ is the homomorphism induced by p. Then, $p_*(\pi_1(M))$ contains the subgroup $\langle \gamma_1^{a_1}, \dots, \gamma_n^{a_n} \rangle$ of $\mathbb{F}_n$. It follows that $p_*(\pi_1(M))$ is free of rank at least n, implying that it admits $\mathbb{F}_n$ as a quotient.

For the converse, every epimorphism $j : \pi_1(M) \to \mathbb{F}_n$ is induced by a simplicial map $g\colon M \to \vee_n S^1$ onto a bouquet of circles, with respect to a suitable combinatorial triangulation of M. There also exists a simplicial embedding $h\colon \vee_n S^1 \to M$ which is a homotopy right inverse of g.

Take a point y_i in the i-th component of $\vee_n S^1 - \{*\}$, where $*$ is the joining point. Up to a small homotopy, we can assume that the points $y_1, \dots, y_n$ are regular values for the map g. Let Y_i be the connected component of $g^{-1}(y_i)$ that contains $h(y_i)$. Then, Y_i is a connected orientable locally flat PL 3-manifold in M.

Let M' be M cut open along $Y_1, \dots, Y_n$. By construction, M' is a connected 4-manifold with $2n$ boundary components $Y_1, \overline{Y}_1, \dots, Y_n, \overline{Y}_n$ and there are identifications $Y_i \cong \overline{Y}_i$ coming from the cuts. By Theorem 10 there exists a simple d-fold branched covering $q\colon M' \to S^4 - \cup_{i=1}^n \operatorname{Int}(B_i^4 \cup \overline{B}_i^4)$ such that the coverings $q_{|Y_i} : Y_i \to \partial B_i$ and $q_{|\overline{Y}_i} : \overline{Y}_i \to \partial \overline{B}_i^4$ match with respect to the above identifications, where B_i^4 and $\overline{B}_i^4$ are disjoint 4-balls in S^4, for $i = 1, \dots, n$. We can assume that B_q is a PL locally flat self-transversally immersed compact surface if $d \geq 4$, and that it is embedded if $d \geq 5$. Then, we can glue back Y_i with $\overline{Y}_i$, as well as ∂B_i^4 with $\partial \overline{B}_i^4$, by means of the identifications needed to reconstruct M and $\#_n(S^3 \times S^1)$ respectively. Then we get a simple branched covering $p\colon M \to \#_n(S^3 \times S^1)$ as desired. □

For the proof of Theorem 8 we need two lemmas.

Lemma 12. *Let $C \subset B^3$ be a properly embedded (not necessarily connected) compact curve. Then, the surface $F = C \times B^1 \subset B^3 \times B^1 \cong B^4$ is ribbon.*

Proof. Up to ambient isotopy, we can assume that the origin $0 \in B^3$ does not belong to C and that the image $D = \pi_0(C) \subset S^2$ of C under the radial projection $\pi_0 : B^3 - \{0\} \to S^2$ from 0 forms only transversal double points (it gives a diagram of C). Let $\pi_{(0,0)} : (B^3 \times B^1) - \{(0,0)\} \to \partial(B^3 \times B^1) = (S^2 \times B^1) \cup (B^3 \times S^0) \cong S^3$ the radial projection from the origin $(0,0) \in B^3 \times B^1$. Then, for each $x \in C$ the image under $\pi_{(0,0)}$ of the segment $\{x\} \times B^1$ is given by $\pi_{(0,0)}(\{x\} \times B^1) = (\pi_0(\{x\}) \times B^1) \cup ([x, \pi_0(x)] \times S^0) \subset \partial(B^3 \times B^1)$, where $[x, \pi_0(x)] \subset B^3$ denotes the segment spanned by x and $\pi_0(x)$. It follows that the image $\pi_{(0,0)}(F) \subset \partial(B^3 \times B^1)$ forms only ribbon intersections, consisting of a single double arc for each double point of D. Hence, F is a ribbon surface. □

Remark 13. In the smooth category, one could argue that the surface F can be realized in B^4 as a ruled surface, not passing through the origin. Then, the distance from the origin restricts to a function on F without local maxima in $\operatorname{Int} F$, which implies that F is ribbon.

Lemma 14. *Let* $N_1, \dots, N_k \subset M$ *be pairwise disjoint compact oriented (locally flat) PL* 3*-manifolds with non-empty boundary, and let* $B^3_1, \dots, B^3_k \subset S^4$ *be pairwise disjoint PL 3-balls. For every* $i = 1, \dots, k$*, let* $c_i \colon N_i \to B^3_i$ *be a simple* d*-fold branched covering, with* $B_{c_i} \subset B^3_i$ *a properly embedded compact curve and* $d \geq 4$*. Then,* $c = c_1 \cup \dots \cup c_k$ *extends to a simple* d*-fold branched covering* $p \colon M \to S^4$ *with* B_p *a locally flat PL surface self-transversally immersed (embedded for* $d \geq 5$*) in* S^4*.*

Proof. We consider pairwise disjoint collars $C_i = C(N_i) \subset M$ of the 3-manifolds N_i in M and pairwise disjoint collars $D_i = C(B^3_i) \subset S^4$ of the 3-balls B^3_i in S^4. Then, we have $C_i \cong N_i \times B^1$ with N_i canonically identified to $N_i \times \{0\}$, and $D_i \cong B^3_i \times B^1$ with B^3_i canonically identified to $B^3_i \times \{0\}$. Up to these identifications and assuming all the collars positively oriented, the branched coverings c_i extend to simple d-fold coverings $c'_i = c_i \times \mathrm{id}_{B^1} : C_i \to D_i$. By Lemma 12, each branch set $B_{c'_i}$ is a ribbon surface in $D_i \cong B^4$. Now, we consider the simple d-fold branched covering $p_1 = \cup_i c'_i \colon \cup_i C_i \to \cup_i D_i$, and put $X = \mathrm{Cl}(M - \cup_i C_i)$ and $Y = \mathrm{Cl}(S^4 - \cup_i D_i)$. The restriction to the boundary of p_1 gives a simple d-fold branched covering $p_{1|\partial} \colon \partial X \to \partial Y$, which is ribbon fillable by construction. Therefore, Theorem 10 allows us to extend $p_{1|\partial}$ to a simple d-fold branched covering $p_2 \colon X \to Y$ with B_{p_2} a locally flat PL surface self-transversally immersed (embedded for $d \geq 5$) in Y. Thus, we can conclude the proof by putting $p = p_1 \cup_\partial p_2$. □

Proof of Theorem 8. Since the normal bundle ν_F is trivial, for every $i = 1, \dots, k$ we can find a 3-dimensional locally flat PL ribbon $N_i \cong F_i \times [0, 1]$ in M such that $\partial N_i = F_i \cup F'_i$, with $F'_i \subset M$ a "parallel" copy of F_i oriented in the opposite way. We assume the N_i's to be pairwise disjoint. Let $N'_i \subset M$ be the 3-manifold obtained by removing the interiors of $d - d_i - 2$ disjoint PL 3-balls from $\mathrm{Int}\, N_i$.

Each surface $\partial N'_i$ admits a d-fold simple branched covering $f_i \colon \partial N'_i \to S^2$, where F_i consists of the sheets 1 to d_i, F'_i consists of the sheets $d_i + 1$ and $d_i + 2$, while the boundaries of the removed 3-balls consists of the remaining $d - d_i - 2 \geq 0$ sheets trivially covering S^2. By Corollary 6.3 in [5], this can be extended to a d-fold simple branched covering $c_i \colon N'_i \to B^3$. After having identified the base spaces of such coverings with a family of disjoint PL 3-balls $B^3_1, \dots, B^3_k \subset S^4$, we can apply Lemma 14 to get a simple covering $p \colon (M; N'_1, \dots, N'_k) \to (S^4; B^3_1, \dots, B^3_k)$ of degree d, branched over a locally flat PL surface self-transversally immersed (embedded for $d \geq 5$) in S^4. Then, p is the desired branched covering, since $p(F) = \partial(\cup_i B^3_i)$ is a trivial link of k spheres.

By the Lüroth-Clebsch theorem (see Berstein and Edmonds [5], or Bauer and Catanese [4] for a different approach), simple branched coverings from a closed connected oriented genus g surface to S^2 are classified by the degree. Then, the restrictions $p_{|F_i}$ can be arbitrarily chosen, up to isotopy, with the given degrees d_i.

If $F_i \cong S^2$ for every i, we set $d_i = 1$ and at the beginning of the proof we remove the interiors of $d - 2$ balls from N_i (instead of $d - 3$) so that N'_i has d boundary components, all homeomorphic to a sphere. Then, by following the same argument, we obtain the desired simple branched covering $p \ : M \to S^4$ such that $T_k \cap B_p = \emptyset$. □

3. Final remarks

In Theorem 1 (a) or (b), the simple branched covering $p\colon M \to CP^2$ can be constructed such that $p^*(w_2(CP^2)) = w_2(M)$. Indeed, in the proof it is enough to take $d(p)$ odd and choose φ' to be a characteristic element, namely the Poincaré dual of an integral lifting of $w_2(M)$.

It is known that there are smooth 4-manifolds $X_{m,n}$ homeomorphic but not diffeomorphic to $\#_m CP^2 \#_n \overline{CP}^2$ for certain $m, n \geq 1$, see for example Donaldson [8], Akhmedov and Park [1, 2] and Park, Stipsicz and Szabó [18]. By Theorem 3 the exotic manifold is a simple branched covering of the standard one, namely there is $p\colon X_{m,n} \to \#_m CP^2 \#_n \overline{CP}^2$.

In Theorem 7 we can take p such that its restriction $p_{|N}$ coincides with any given ribbon fillable d-fold branched covering $c\colon N \to S^3$. Indeed, in the proof the choice of c as such a covering is arbitrary.

The following Corollary to Theorem 7 is immediate but possibly interesting for the PL Schoenflies Conjecture in S^4.

Corollary 15. *Let $\Sigma^3 \subset S^4$ be a PL embedded 3-sphere and let $d \geq 4$. Then, there exists a d-fold simple covering $p\colon (S^4; \Sigma^3) \to (S^4; S^3)$ branched over a locally flat PL self-transversally immersed surface, which can be taken embedded for $d \geq 5$. Moreover, the restriction $p_{|\Sigma^3}\colon \Sigma^3 \to S^3$ can be arbitrarily chosen among d-fold ribbon fillable branched coverings.*

Moreover, for a PL 3-manifold $N \subset M$, one can prove that there is a simple branched covering $p\colon (M; N) \to (S^4; S^3)$ even though N does not disconnect M. In this case, we obtain an arbitrary degree $d \geq 6$ and a locally flat PL embedded branch surface. The proof goes as follows: following the proof of Theorem 7 (b), we begin with a ribbon fillable branched covering $c\colon \partial M_1 \to S^3$ of degree $d \geq 6$, with M_1 a collar of N in M. This is possible because ∂M_1 has two connected components homeomorphic to N. Then, by Theorem 10, there are two extensions of c as simple d-fold branched coverings $p_1 : M_1 \to S^4_-$ and $p_2 : M_2 \to S^4_+$, both branched over a PL locally flat properly embedded surface. Their union provides the desired branched covering $p\colon (M; N) \to (S^4; S^3)$.

In Theorem 8, for $k \geq 2$, we can take $S^2 \subset S^4$ instead of T_k, with $d = 4k$. Thus, there exists a simple branched covering $p\colon (M; F) \to (S^4; S^2)$ even though F is not connected. The proof is essentially the same, the only difference consisting in the identification of the base of $c_i : N_i \to B^3$ with a single copy of $B^3 \subset S^4$ instead of k copies of it.

The singularities of the branch surfaces of all the 4-dimensional simple branched coverings we have constructed, namely the transversal self-intersections, originate from the application of Theorem 10, which was proved in [20]. In the construction therein, such singularities appear in pairs, so one can investigate to what extent they can be eliminated, without increasing the covering degree. Then, we conclude by asking the following question (cf. Problem 4.113 (A) in Kirby's list [13]).

Question 16. *Can the simple branched covering $p\colon M \to N$ in Theorem 1 be always chosen with a locally flat PL embedded branch surface even for $d = 4$?*

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